SELECTIVE CONTROL OF INDEPENDENT ACTIVITIES:
LINEAR PROGRAMMING OF MARKOVIAN DECISIONS+

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Abstract

This study is related to optimum control strategies used by a manager supervising a group of independent activities whose performances deteriorate with time. At the end of a day, the manager evaluates information regarding the performance levels of the activities during that day and decides which activity to attend to, since he can control only one at a time. The deterioration of performance and the improvement of the deteriorated performance by a control action from one level to other levels are given by stationary transition probabilities. Optimal strategies for this control are obtained by the linear programming method proposed by Manne [7].

I. Introduction

The present paper studies a case where a manager controls a group of independent activities whose performances deteriorate with time. This management control is presumed to be an iterative process composed of three steps: (1) the observation of the conditions of activities, (2) the selection of an optimal course of action, and (3) the execution of this course of action.

Each of the activities may be considered as a man, a machine, or a man-machine system and is engaged in production of goods or services independent of the other activities. It consumes direct labor and materials at constant rates. Its performance is measured by the net utility produced during the day and represented by one of a finite number of "performance levels" or merely "levels". The performance of each activity tends to deteriorate with time; the transition of the activity from a level in the previous day to various other levels in the following day is given by a set of stationary probabilities. The stationary transition probabilities mean that the basic characteristics of the activity do not change in time.

The manager is responsible for maintaining the performance of each activity at a high level by executing proper control actions for the activity whenever necessary. At the end of a day he evaluates the performance of each activity during that day and determines whether a control action is necessary to improve the deteriorated performance. The execution of the control action is carried out in two steps: First, the manager examines the conditions of the activity's essential factors

attributable to the existing performance level. Then, on the basis of this examination, he selects and applies a control action to the activity. The length of time required to complete the control action depends on the activity and its current performance level. Although his control action aims at the best result possible for the existing level, it does not necessarily achieve this result, owing mainly to inaccuracy in his examination of the conditions and execution of corrective measures. Therefore, the result of a control action depends on the existing performance level of the activity and is given not by a single improved level but by a set of levels with a probability distribution, essentially reflecting the skill of the manager. This probability distribution is assumed stationary, which means the skill of the manager neither improves nor erodes with time.

One important restriction imposed on the manager is that he can attend to only one activity for control at one time. Under this restriction, he gives attention simultaneously to all the activities and determines not just whether these activities need control actions but also, should more than one of them need control actions, which one has priority. This requires him to set up a priority rule of control in each "state" of the system, or each of the combinations of the possible performance levels of the activities. His objective is the maximization of the sum of the average net utilities produced per day by the individual activities, the values of individual production less the costs of control actions representing the only type of cost considered in the analysis. Solutions to optimal strategies for control are obtained by using the method of linear programming for sequential decisions proposed by A. S. Manne [7],

under the assumption of a Markovian steady state for the operation of the system. Wagner [9] proved the existence of an optimal solution having only pure strategies in this method.

The steady state is derived from the stationary transition probabilities that have been assumed for the performance deterioration and improvement of activities. Although in theory the steady state distribution can be reached from any initial distribution after an infinite number of transitions, in reality an ergodic chain produces an approximately constant distribution, a steady state in the sense of practical use, in a finite number of transitions and often in only several transitions. Manne indicated the omission of initial conditions and time discounting as shortcomings of his method. The omission of initial conditions for Markov process may significantly affect the selection of optimal strategies; specifically, if the optimal matrix in the linear programming solution is a "decomposable" one, the initial conditions will clearly govern the ultimate statistical equilibrium [7, p. 264]. Later, D'Epenoux [8] and de Ghellinck and Eppen [3] proposed linear programming methods that take into consideration both the initial conditions and time discounting.

Derman [4] and Klein [6] discussed problems related to the control of a deteriorating system with the assumption that the time between successive transitions depended on the control decision. They achieved the minimization of the average cost per unit time represented by an objective function in fractional form through the transformation of the original formulation to the linear programming formulation of Manne. Using their method we transform our original formulation with a fractional objective function to a linear programming formulation.

II. Discussion

The organization being studied here is composed of a manager and a system of two independent, dissimilar activities, Activity I and Activity II. At the end of a day the manager collects information regarding the conditions of the activities during the day and, on the basis of this information, determines whether Activity I or Activity II should be given a control action.

1. Performance Levels, Deterioration, and Improvement

The condition of each activity at the end of the day is described by one of a finite number of performance levels, $L_1, \ldots, L_m$ for Activity I and $L_1^*, \ldots, L_{m^*}^*$ for Activity II. (Whenever a pair of identical symbols is used, a symbol without * pertains to Activity I and a symbol with * to Activity II.)

The utilities produced at these levels are $v_1, \ldots, v_m$ for Activity I and $v_1^*, \ldots, v_{m^*}^*$ for Activity II and given by vectors V and V*, respectively.

$$(1) \qquad V = \begin{bmatrix} v_1 \\ \cdot \\ v_m \end{bmatrix}, \qquad V^* = \begin{bmatrix} v_1^* \\ \cdot \\ v_{m^*}^* \end{bmatrix}$$

The utilities within each set have no identical values and can be ordered by preference as follows:

$$(2) \qquad v_1 > \ldots > v_m \geqq 0, \qquad v_1^* > \ldots > v_{m^*}^* \geqq 0$$

If the utility produced at the lowest level v_m or $v_{m^*}^*$ equals zero, this level represents the condition of breakdown, thereby producing no utility.

Natural deterioration of the performance level of each activity is a Markov process in which the activity daily moves from one level to other levels with given probabilities that stay stationary over time. These transition probabilities for Activity I are represented by $p_{i,h}$ with i being the level on the preceding day and h the level on the following day; similarly, $p^*_{j,k}$ denotes these probabilities for Activity II. (Symbols with double subscripts separated by a comma, such as $p_{i,h}$, are used when the event shown by the second subscript is conditional to the event shown by the first subscript. On the other hand, double subscripts without a comma, as with X^z_{ij} or d^z_{ij} to be introduced later, denote both events occur simultaneously.) Probabilities $p_{i,h}$ and $p^*_{j,k}$ are given as elements of vector p_i and p^*_j, respectively.

(3) $\quad p_i = (p_{i,1}, \ldots, p_{i,m}) \qquad i=1,\ldots,m$

where

$$p_{i,h} \geq 0, \qquad \sum_{h=1}^{m} p_{i,h} = 1$$

Similarly, we have

(4) $\quad p^*_j = (p^*_{j,1}, \ldots, p^*_{j,m^*}) \qquad j=1,\ldots,m^*$

where

$$p^*_{j,k} \geq 0, \qquad \sum_{k=1}^{m^*} p^*_{j,k} = 1$$

The assumption that the performance of an activity does not improve without

a control action means that $p_{i,h}$ $(h=1,\ldots,i-1)$ and $p^*_{j,k}$ $(k=1,\ldots,j-1)$ are all zeros.

The length of time required for completing a control action depends on the activity and its current performance level. The result of the control action also is a Markov process in which the transition of the activity moves from one level to improved levels is given by a set of stationary probabilities. Let $q_{i,h}$ be the probability with which the performance of Activity I improves from level L_i to level L_j because of a control action; likewise, let $q^*_{j,k}$ be this probability for Activity II. These transition probabilities are given by vectors q_i and q^*_j.

(5) $$q_i = (q_{i,1},\ldots,q_{i,m}) \qquad i=1,\ldots,m$$

where

$$q_{i,h} \geqq 0, \quad \textstyle\sum_{h=1}^{m} q_{i,h} = 1$$

Similarly, we have

(6) $$q^*_j = (q^*_{j,1},\ldots,q^*_{j,m^*}) \qquad j=1,\ldots,m^*$$

where

$$q^*_{j,k} \geqq 0, \quad \textstyle\sum_{k=1}^{m^*} q^*_{j,k} = 1$$

If a control action never lowers the performance of an activity, $q_{i,h}$ $(h=i+1,\ldots,m)$ and $q^*_{j,k}$ $(k=j+1,\ldots,m^*)$ are all zeros. Further, if $q_{i,i} > 0$ or $q^*_{j,j} > 0$, the control action executed at L_i or L^*_j may fail to improve the performance of the activity. This implies that the time

required for improving the performance to a higher level is uncertain, although a constant time is required for executing a control action.

2. States and Expected Utilities

Clearly, the performance levels of the two activities during the day determine the next course of action to be selected by the manager. Since there are m possible levels for Activity I and m^* possible levels for Activity II, there are mm^* possible combinations of the levels of these activities. We name each of the combinations a "state" of the system and denote it by S_{ij} where i represents the level for Activity I and j the level for Activity II. In each state the manager may take a particular course of action, z, from the set Z of alternative courses of action:

$$z \varepsilon Z = \{0, 1, 2\}$$

where

z = 0: To take no control action on either activity.

(7) z = 1: To take an action on Activity I but not on Activity II.

z = 2: To take an action on Activity II but not on Activity I.

Both activities are engaged in production of independent items and the group utility is the sum of their net utilities, the utilities produced less the costs of control. It takes one day for the manager to collect information regarding the performance of each activity. Let r_i and c_i be the number of days and the cost required for the execution of a control action on Activity I at L_i; r_j^* and c_j^* be those required for the execution of a control action on Activity II at L_j^*; and d_{ij}^z be the number of

days between the decision to take control action z in S_{ij} and the next decision point. If the manager is constantly engaged in either collecting information or executing a control action, d^z_{ij} is given as follows:

$$(8) \qquad d^z_{ij} = \begin{cases} 1 & \text{if } z = 0 \\ r_i+1 & \text{if } z = 1 \\ r^*_j+1 & \text{if } z = 2 \end{cases}$$

An activity produces no utility while it is given a control action, whereas the other activity continues its production. Further, any decision is irreversible and a new decision cannot be made until the time set by the last decision. The net utility expected of the system during the period d^z_{ij} is given by the following W^z_{ij}:

$$(9) \qquad W^0_{ij} = p_i V + p^*_j V^*$$

$$(10) \qquad W^1_{ij} = \{q_i V + p^*_j(I+P^*+ \dots +P^{*\,r_i})V^* - c_i\}$$

$$(11) \qquad W^2_{ij} = \{p_i(I+P+ \dots + P^{r^*_j})V + q^*_j V^* - c^*_j\} \qquad i=1,\dots,m;\ j=1,\dots,m^*$$

where P and P* are the following matrices composed of row vectors p_j and p^*_j:

$$(12) \qquad P = \begin{bmatrix} p_1 \\ \cdot \\ p_m \end{bmatrix}, \qquad P^* = \begin{bmatrix} p^*_1 \\ \cdot \\ p^*_{m^*} \end{bmatrix}$$

3. Formulation of a Linear Program

Let x^z_{ij} be the state variable representing the proportion of the time in which the course of action z is selected in state S_{ij}. Then the

following function gives the average net utility per day of the system which the manager wants to maximize:

$$(13) \qquad F = \frac{\sum_{z\varepsilon Z} \sum_{i=1}^{m} \sum_{j=1}^{m^*} W_{ij}^{z}\ x_{ij}^{z}}{\sum_{z\varepsilon Z} \sum_{i=1}^{m} \sum_{j=1}^{m^*} d_{ij}^{z}\ x_{ij}^{z}}$$

subject to

$$(14) \qquad x_{ij}^{z} \geq 0 \qquad\qquad Z = \{0,\ 1,\ 2\}\quad i=1,\ldots,m;\ j=1,\ldots,m^*$$

Further, since the state variables are probability variables, they must satisfy the following unitary constraint:

$$(15) \qquad \sum_{z\varepsilon Z} \sum_{i=1}^{m} \sum_{j=1}^{m^*} x_{ij}^{z} = 1$$

In steady state, the probability of getting into a state is equal to the probability of getting out of this state. This statistical equilibrium exists for each of mm^* states S_{hk} $(h=1,\ldots,m;\ k=1,\ldots,m^*)$ and is given by

$$(16) \qquad \sum_{i=1}^{m}\sum_{j=1}^{m^*} \{p_{i,h}p_{j,k}^{*}\ x_{ij}^{0} + q_{i,h}p_{j,k}^{*\,(r_i+1)}\ x_{ij}^{1} + p_{i,h}^{(r_j^*+1)}\ q_{j,k}^{*}x_{ij}^{2}\}$$

$$- \sum_{z\varepsilon Z} x_{hk}^{z} = 0 \qquad\qquad Z = \{0,\ 1,\ 2\}\quad h=1,\ldots,m;\ k=1,\ldots,m^*$$

where

$p_{i,h}^{(r_j^*+1)}$ = the i-hth entry of matrix $P^{r_j^*+1}$

$p^{*(r_i+1)}_{j,k}$ = the j-kth entry of matrix P^{r_i+1}

Following Derman [4, p. 22] and Klein [6, p. 31], we now reformulate the above non-linear objective function (13) in x^z_{ij} to a linear function in a new variable, X^z_{ij}, defined as

(17) $$X^z_{ij} = \frac{x^z_{ij}}{T} \qquad z\varepsilon Z; \quad i=1,\ldots,m;\ j=1,\ldots,m^*$$

where

(18) $$T = \sum_{z\varepsilon Z}\sum_{i=1}^{m}\sum_{j=1}^{m^*} d^z_{ij}\ x^z_{ij}$$

Using X^z_{ij}, we rewrite the objective function (13) as

(19) $$F = \sum_{z\varepsilon Z}\sum_{i=1}^{m}\sum_{j=1}^{m^*} W^z_{ij}\ X^z_{ij}$$

Since T is positive and x^z_{ij} is non-negative, we have

(20) $$X^z_{ij} \geq 0 \qquad z\varepsilon Z; \quad i=1,\ldots,m;\ j=1,\ldots,m^*$$

From (17) and (18), we get the following unitary constraint:

(21) $$\sum_{z\varepsilon Z}\sum_{i=1}^{m}\sum_{j=1}^{m^*} d^z_{ij}\ X^z_{ij} = 1$$

Finally, (16) is transformed to the following equation retaining the original coefficients:

$$(22)\quad \sum_{i=1}^{m}\sum_{j=1}^{m^*}\{p_{i,h}p^*_{j,k}X^0_{ij} + q_{i,h}p^{*(r_i+1)}_{j,k}X^1_{ij} + p^{(r^*_j+1)}_{i,h}q^*_{j,k}X^2_{ij}\}$$

$$- \sum_{z\varepsilon Z}X^z_{hk} = 0 \qquad\qquad Z = \{0,\ 1,\ 2\}$$
$$h=1,\dots,m;\ k=1,\dots,m^*$$

Equations (19)-(22) are all linear functions in X^z_{ij}. There are mm^* equations in (22), any one of which can be derived from the rest. Against mm^* states, we have the same number of independent equations in (21) and (22), which guarantees pure strategies giving a positive value to only one of the state variables related to each state [9, p. 268]. With (19)-(22), we have completed the formulation of a linear program.

To find an optimal solution of x^z_{ij} to the original problem, first we obtain T from (15) and (17):

$$(23)\quad T = \frac{1}{\sum_{z\varepsilon Z}\sum_{i=1}^{m}\sum_{j=1}^{m^*}X^z_{ij}}$$

With this T and the transformation (17), we may find an optimal value of x^z_{ij} for the optimal X^z_{ij} determined by the above linear program.

III. Numerical Example

In this example both Activity I and Activity II have six possible performance levels with utilities given by the following vectors V and V*:

$$V = (50 \quad 47.5 \quad 45 \quad 40 \quad 35 \quad 30)^T, \qquad V^* = (45 \quad 40 \quad 35 \quad 30 \quad 25 \quad 20)^T$$

Transition probabilities for performance deterioration are given by matrix P and vector p_i $(i=1,\ldots,6)$ for Activity I and matrix P* and vector p_j^* $(j=1,\ldots,6)$ for Activity II.

$$P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} .94 & .03 & .02 & .01 & .00 & .00 \\ & .86 & .05 & .04 & .03 & .02 \\ & & .82 & .10 & .05 & .03 \\ & & & .79 & .15 & .06 \\ & 0 & & & .76 & .24 \\ & & & & & 1.00 \end{bmatrix}, \quad P^* = \begin{bmatrix} p_1^* \\ p_2^* \\ p_3^* \\ p_4^* \\ p_5^* \\ p_6^* \end{bmatrix} = \begin{bmatrix} .97 & .02 & .01 & .00 & .00 & .00 \\ & .94 & .03 & .02 & .01 & .00 \\ & & .92 & .04 & .03 & .01 \\ & & & .90 & .06 & .04 \\ & 0 & & & .90 & .10 \\ & & & & & 1.00 \end{bmatrix}$$

Since L_6 and L_6^* are the poorest levels, both $p_{6,6}$ and $p_{6,6}^*$ are equal to 1.

Similarly, transition probabilities for performance improvement owing to control actions are given by matrix Q and vector q_i $(i=1,\ldots,6)$ for Activity I and matrix Q* and vector q_j^* $(j=1,\ldots,6)$ for Activity II.

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 1.00 & & & & & \\ .80 & .20 & & & & \\ .60 & .30 & .10 & & 0 & \\ .30 & .40 & .22 & .08 & & \\ .20 & .30 & .26 & .18 & .06 & \\ .10 & .22 & .30 & .20 & .12 & .06 \end{bmatrix}, \quad Q^* = \begin{bmatrix} q_1^* \\ q_2^* \\ q_3^* \\ q_4^* \\ q_5^* \\ q_6^* \end{bmatrix} = \begin{bmatrix} 1.00 & & & & & \\ .82 & .18 & & & & \\ .68 & .22 & .10 & & 0 & \\ .50 & .28 & .14 & .08 & & \\ .34 & .32 & .18 & .13 & .03 & \\ .22 & .32 & .22 & .16 & .06 & .02 \end{bmatrix}$$

Since levels L_1 and L_1^* cannot be improved, $q_{1,1}$ of Q and $q_{1,1}^*$ of Q^* are both equal to 1.

Costs of control actions are given by the following vectors C and C^*:

$$C = (20 \quad 21 \quad 22 \quad 23 \quad 24 \quad 25)^T, \quad C^* = (25 \quad 26 \quad 27 \quad 28 \quad 29 \quad 30)^T$$

We assume the execution of a control action takes one day for either activity at any level: $r_i = r_j^* = 1 \cdot (i, j=1,\ldots,6)$. Therefore, the numbers of days between successive decision points are given by

$$d_{ij}^0 = 1, \; d_{ij}^1 = d_{ij}^2 = 2$$

A computer run of this program took 33 seconds on an IBM 7094, producing the optimal solution of X_{ij}^z listed in Table 1. From these X_{ij}^z's, we find T = 1.1246 and the expected net utility F = 86.4. From this T and the transformation (17), we have obtained optimal values of the original variables x_{ij}^z's corresponding to the optimal X_{ij}^z's, as is shown in Table 1.

The optimal rules obtained above are graphically shown in Figure 1 indicating a particular course of action the manager should take in each of the states. For example, if the state of the system is S_{24}, the crosspoint a_{24} of the horizontal line drawn through L_2 and the vertical line drawn through L_4^* falls in Region III, instructing the manager to take a control action for Activity II.

We have tried several computer runs and found that once transition probability matrices P, P^*, Q, and Q^* are fixed, optimal strategies in various states are relatively insensitive to changes in utilities given by V and V^* or changes in costs of control actions given by C and C^*.

Table 1. Result of Computer Run--Optimal Courses of Action Selected for Various States and Probabilities of Systems Getting Into These States

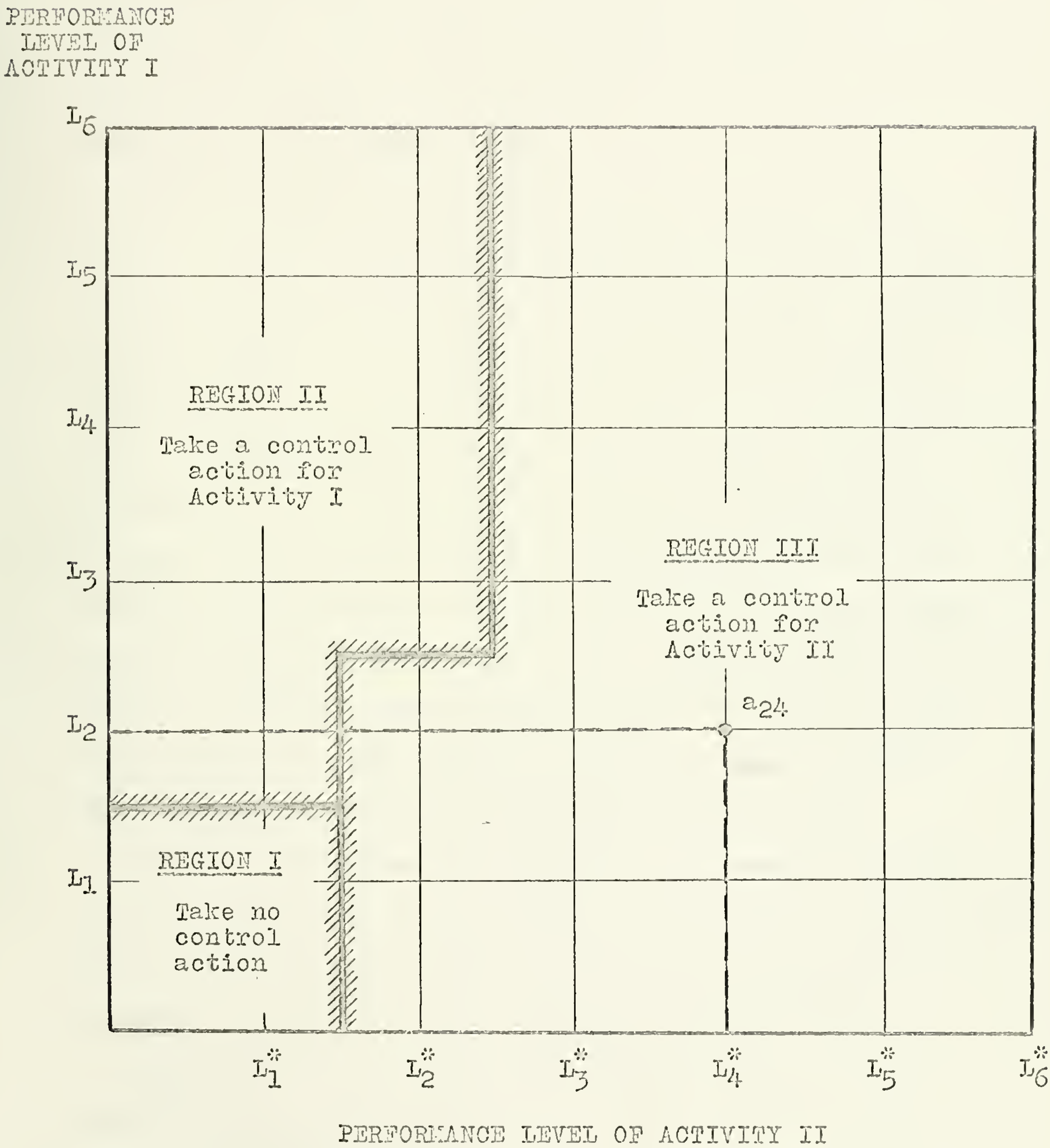
States S_{ij}	Optimal Courses of Action z	New Variables X_{ij}^z	Probabilities (Original Variables) $x_{ij}^z = TX_{ij}^z$	Net Utilities Per Day W_{ij}^z	States S_{ij}	Optimal Courses of Action z	New Variables X_{ij}^z	Probabilities (Original Variables) $x_{ij}^z = TX_{ij}^z$	Net Utilities Per Day W_{ij}^z
S(1,1)	0	.778368	.875373	94.5	S(4,1)	1	.009378	.010547	113.4
S(1,2)	2	.023464	.026388	117.2	S(4,2)	1	.000405	.000455	102.6
S(1,3)	2	.009884	.011116	115.0	S(4,3)	2	.000011	.000011	92.0
S(1,4)	2	.000019	.000021	112.1	S(4,4)	2	.000001	.000001	89.0
S(1,5)	2	.000010	.000011	109.2	S(4,5)*	2	ε	ε	86.1
S(1,6)*	2	ε	ε	106.2	S(4,6)*	2	ε	ε	83.2
S(2,1)	1	.043566	.048995	117.9	S(5,1)	1	.000134	.000151	110.6
S(2,2)	2	.001796	.002020	109.6	S(5,2)	1	.000030	.000034	99.7
S(2,3)	2	.000464	.000522	107.4	S(5,3)	2	.00007	.000008	82.6
S(2,4)	2	.000017	.000019	104.5	S(5,4)	2	.000001	.000001	79.7
S(2,5)	2	.000011	.000012	101.6	S(5,5)*	2	ε	ε	76.7
S(2,6)*	2	ε	ε	98.6	S(5,6)*	2	ε	ε	73.8
S(3,1)	1	.020844	.023442	116.1	S(6,1)	1	.000101	.000114	107.3
S(3,2)	1	.000612	.000688	105.2	S(6,2)	1	.000028	.000031	96.4
S(3,3)	2	.000021	.000024	101.6	S(6,3)	2	.000006	.000007	75.9
S(3,4)	2	.000005	.000006	98.7	S(6,4)*	2	ε	ε	73.0
S(3,5)	2	.000001	.000001	95.8	S(6,5)*	2	ε	ε	70.0
S(3,6)*	2	ε	ε	92.8	S(6,6)*	2	ε	ε	67.1

$T = 1 \Big/ \sum_{z\varepsilon Z}\sum_{i=1}^{6}\sum_{j=1}^{6} X_{ij}^z = 1.124626$

Expected Net Utility Per Day $F = \sum_{z\varepsilon Z}\sum_{i=1}^{6}\sum_{j=1}^{6} W_{ij}^z X_{ij}^z = 86.4$

Note *: Although the courses of action indicated are selected in these states, the values of X's are less than .000001 and not printed out.

Figure 1. Optimal Courses of Action in Various States



IV. Summary

Management control of a group of independent dissimilar activities has been studied in this article. This control is presumed to be an iterative process composed of three steps--daily observation of the conditions of activities, determination of an optimal course of action, and execution of this course of action.

The study assumes that the performance of each activity tends to deteriorate, and that the deteriorated performance can be improved only by a control action applied to the activity by the manager. One important restriction imposed on the manager is that he may control only one activity at a time. This requires him to determine which activity has priority, should both activities need control actions. Further, the study assumes that the performance deterioration and improvement from one level to other levels are given by stationary probabilities. The assumptions of the stationary probabilities imply that neither a change in the characteristics of the activity nor a change in the skill of the manager is considered. These assumptions enable us to obtain a steady state solution for optimal control strategies by using the method of linear programming for sequential decisions proposed by Manne.

A numerical example has been illustrated. The results of examples not reported show that, once the transition probabilities for deterioration and improvement are fixed, optimal strategies are relatively insensitive to changes in utilities or costs of control in various states.

Although the transition period used for deterioration matrices P and P^* is one day, a longer period may be feasible when the rate of performance deterioration is slow. If the period is changed to two

days, three days, etc., matrices P and P^* should be replaced by P^2 and P^{*2}, P^3 and P^{*3}, etc., accordingly.

The organization studied here is of the simplest type having only two independent activities. Further the decision involves only whether or not activities should be given control actions at given performance levels. With minor changes in its basic structure, the present formulation can readily be extended to an organization composed of N activities and the decision by the manager can be broadened to consider alternative control actions available at each level of an activity, any number of time intervals to elapse between successive decision points, or variable periods required between the time of observing the performances of activities and the time of making a decision. With such modifications, the model formulated here would better approximate the process of management control in some real-world situations where the service of a common apparatus is essential for maintaining the operations of many activities. Illustrations of such situations include cases where one operator runs many machines, a crew of mechanics maintains a fleet of cars, and a computer center processes information required for the operations of various plants.

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